MODELS FOR ELASTIC DEFORMATION OF AUXETIC HONEYCOMB WITH TRIANGULAR CELLS

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Honeycomb, Auxetic, elastic, Properties, Model

Abstract
Studies devoted to honeycombs are aimed at improvement of their mechanical properties. One of the methods is a core structure modification by adding auxetic features. Therefore, for modelling, it is relevant and necessary to describe the deformation mechanism and honeycomb elastic constants. In the case of hexagonal cellular structures, this field is well recognized. On the other hand, there is little research dedicated to the analytical description of a deformation form in the context of material properties of non-hexagonal auxetic honeycombs. The aim of the study was to develop a theoretical model for predicting triangular auxetic core elastic constants based on cell deformation by pure flexure. Axial displacements of cell nodes were established as well as expressions for moduli of elasticity, shear moduli and Poisson's ratios. Derived dependencies showed how the material properties and the applied load direction affected the deformation mechanism of an auxetic honeycomb with triangular cells.

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1. INTRODUCTION

Cell panels, due to their capability to absorb energy as well as their high ratio of strength to mass (Becker 1998; Meraghni et al., 1999; Pan et al., 2008) are employed in: motor industry (chassis construction, brake disks, car bodies, fender undersides); airplanes (skin of the fuselage and wings); in shipping industry (side plating of boats, yachts, ships); the construction of spaceships, satellites, space probes, tanks and containers; armaments industry (rocket construction) (Aydincak 2007). In the case of the above-mentioned constructions, the core of cell panels is, most frequently, manufactured from aluminum. Together with the development of our knowledge about cellular structures with traditional, hexagonal cores, many researchers developed analytical descriptions of their mechanical properties. Pioneering experiments in this field were conducted by McFarland (1963, 1964). Wierzbicki (1983) proposed an analytical model of a quasi static strength and energy dissipation during crashing of a hexagonal insert of a cell panel. This approach was improved in some later papers. Said and Tan (2008) demonstrated that the obtained results were underestimated by approximately 20%. Sadighi and Mahmoudabadi (2009) put forward an improved calculation model which takes into consideration the impact of curvature of the constituent fragments of the loaded cell wall and, in this way, leads to improvements in the quality of the obtained results.

Core mechanical properties of cell panels are influenced very strongly by: the length and inclination angle of cell arms, thickness of cell walls, apparent density and the kind of the applied material (Chang i Ebcioglu 1961, Alqassim 2011; Balawi and Abot 2008; Chi et al., 2010; Chen and Pugno, 2012; He and Hu 2008; Hu et al., 2013; Jeyakrishnan et al., 2013; Schultz et al., 2012). In their article, Meraghini et al. (1999) put forward a calculation model
which takes into account combination of the core elastic properties with its geometry and the physico-chemical properties of the material it is made of. Many researchers investigated models which make it possible to predict the range of energy absorption and to determine the form of deformations of multilayer systems (Jeyasingh 2005; Paik et al., 1999; Zhu and Chen 2010; Cote et al., 2004; Dapeng et al., 2011; Crupi et al., 2012). It is evident from the literature on the subject that one of the methods of improving mechanical properties is modification of core geometry (Jeyakrishnan et al., 2013; Yamashita and Gotoh 2005; Chen and Pugno 2012; Schultz et al., 2012). In the context of this change, it turned out very effective to bring the core shape to auxetic form. The term “auxetic” was introduced in 1991 by Professor Kenneth E. Evans from Exeter University who adapted a Greek word “auxeticos” which means “the one that can be made bigger”. In contrast to standard materials, dimensions of the cross section of an auxetic object diminish in the course of a compression test and increase during the axial stretching test (Evans and Alderson 2000). These are materials with a considerable application potentials (Lakes and Witt 2000; Liu and Hu 2010; Yang et al., 2004; Stavroulakis 2005; Pasternak and Dyskin 2012, Critchley et al., 2013). It was demonstrated that they improved: panel resistance to denting (Evans and Alderson 2000; Grima et al., 2012a; Milller et al., 2011); linear elasticity modulus as well as bending strength (Assidi and Ganghoffer 2012; Donescu et al., 2009; Gaspar et al., 2005; Grima et al., 2013; Ju and Summers 2011; Prawoto 2012; Smardzewski 2013) and also the ability to absorb impact energy (Ju and Summers, 2011; Scarpa et al., 2003a) or more effective sound absorption (Scarpa et al., 2003). Elastic properties of a special instance of auxetic structures, i.e. chiral forms (Prall and Lakes 1996) were also elaborated analytically and presented in a number of papers (Alderson et al., 2010; Scarpa et al., 2007; Lorato et al., 2010). In each example, the analytical model provided a point of departure to the description of the effects of a developing deformation. Improvements introduced by successive researchers led to obtaining results of better quality.

There are only very few articles in available literature dealing with auxetic structures in the context of furniture cell panels with a paper core (Smardzewski 2013; Smardzewski and Majewski 2014). Majority of these papers are based on the core of hexagonal geometry of a single cell.

In this study, the authors decided to present an analytical model of a core with non-hexagonal, auxetic cells. So far, we described only geometrical relationships of their parameters occurring in the process of an advancing deformation (Smardzewski and Majewski 2014). These experiments were carried out on the basis of a classical theory of beam mechanics, similar to the conspicuous mathematic description presented by Masters and Evans (1996). As a result of the performed analyses, equations of axial displacements and basic elastic properties of the core were determined taking into account its anisotropic features: Young’s modulus E, Kirchoff’s modulus G and Poisson’s coefficient. The course of changes of their values in the function of the inclination angle of the cell arms was presented. The derived dependencies showed how the material properties and the applied load direction affected the deformation mechanism of an auxetic honeycomb with triangular cells. On the basis of the mathematical model, preferred values of the cell parameters will be selected for further investigations.
Elasticity of hexagonal and auxetic cells

There are many analytical descriptions in literature of elastic properties of thin-walled cell structures characterised by a negative Poisson's coefficient (Grima et al., 2011; Smardzewski 2013; Rad et al., 2014). Among the basic descriptions are those presented in the paper published by Gibson and Ashby (1988). Masters and Evens (1996) expanded them and elaborated a global model which includes the results of action of three deformation mechanisms: bending, compressing and collapsing of cell walls. In analyses they considered hexagonal cell geometry in a traditional and auxetic system (Fig. 1).

Interrelationships occurring in geometry were considered in the context of characteristic parameters: h – width of the cell arm, l – length of the cell arm, θ – inclination angle of arms. Next, the authors used a smart procedure which consisted in the introduction into equations a dependence comprising the value of the acting load and which was a result of this displacement – stiffness coefficient K which was appropriately selected for each of the three kinds of deformations:

- stiffness coefficient $K_r$ at bending:
  \[ K_r = \frac{E_s b l^3}{3 t^3} \left[ \frac{N}{mm} \right], \]  
  (1)

- stiffness coefficient $K_s$ at stretching:
  \[ K_s = \frac{E_s b t}{l} \left[ \frac{N}{mm} \right], \]  
  (2)

- stiffness coefficient $K_h$ for cell wall collapse:
  \[ K_h = \frac{E_s b l^3}{6 t q} \left[ \frac{N}{mm} \right], \]  
  (3)

where:

$E_s$ - Young’s modulus of the cell wall material [MPa],

b – height of the core cell [mm],

t – thickness of the core cell [mm],

q – beam length [mm] as affected by the local displacement of the arm of the cell.

The ultimate elastic properties of cells were described by the following equations:

- Young’s modulus $E$ for anisotropy direction 1:
The presented study aims at determining elastic properties of the auxetic cell treated as a system subjected to pure bending.

**Elasticity of triangular-shaped auxetic cells**

**Cell geometry**

The shape of the new auxetic cell was inspired by similar forms illustrated in a number of publications (Xu et al., 2001; Larsen et al., 1997; Gilat and Aboudi 2013) (Fig. 2).

Using a mathematical model, the authors decided to show how geometrical changes affected elastic properties of a new auxetic system subjected to bending (Fig. 3). Half of the cell symmetrical geometry made up of triangles was employed. It was assumed that desirable displacements of node geometry would be determined in the direction of X and Y axes following the application of P forces. A comprehensive description of the presented diagram takes into account the following parameters: a, b – length of arms (const.) [mm]; h – half length of the cell [mm]; l – cell length [mm]; c, d – dimensions changing their values depending on progress of the cell deformation [mm]; α – internal inclination angle of cell arms; α – angle contained between the longer arm of the cell (b) and the auxiliary axis, parallel to axis X; φ - angle contained between the shorter arm of the cell (a) and the auxiliary axis, parallel to axis Y.

\[ E_1 = \frac{1}{\frac{b \cos \theta}{l} \cos^2 \theta + \frac{K_f}{K_h} + \frac{K_f}{K_s} \frac{h \sin \theta}{l} + \frac{K_s}{K_h}} \text{[MPa]} \]  

- Young's modulus E for anisotropy direction 2:

\[ E_2 = \frac{1}{b \left[ \frac{h}{l} + \sin \theta \right] \left[ \frac{\sin^2 \theta}{K_f \cos^2 \theta} + \frac{\sin^2 \theta}{K_h \cos^2 \theta} + \cos \theta \right] \text{[MPa]} \]  

- Poisson's coefficient ϑ in plane 12:

\[ \dot{\theta}_{12} = -\sin \theta \left( \frac{h}{l} + \sin \theta \right) \left[ \frac{1}{K_f} \frac{1}{K_h} + \frac{1}{K_s} \frac{2h \sin^2 \theta}{K_f \cos^2 \theta} + \frac{2h \sin^2 \theta}{K_h \cos^2 \theta} \right] \frac{1}{\frac{\sin^2 \theta}{K_f \cos^2 \theta} + \frac{\sin^2 \theta}{K_h \cos^2 \theta} + \cos \theta} \]  

- Poisson's coefficient ϑ in plane 21:

\[ \dot{\theta}_{21} = -\sin \theta \cos \theta \left( \frac{1}{K_f} \frac{1}{K_h} + \frac{1}{K_s} \frac{h}{l} + \sin \theta \right) \frac{1}{\frac{\sin^2 \theta}{K_f \cos^2 \theta} + \frac{\sin^2 \theta}{K_h \cos^2 \theta} + \cos \theta} \]  

The presented study aims at determining elastic properties of the auxetic cell treated as a system subjected to pure bending.

Figure 2. Four-nodal, symmetrical geometry of the auxetic cell.
Displacement $\delta_x$

It was necessary to analyze displacement of cell nodes in the direction of axis X and axis Y caused by the application of $P$ forces. On the basis of the adopted geometry, the following equations of parameters were determined:

$$l = \sqrt{b^2 - h^2}, h = \cos \varphi,$$

$$c = \sqrt{a^2 + b^2 - 2ab \cos [90 - \alpha - \varphi]}, \alpha = \arcsin \left(\frac{a}{b} \cos \varphi\right), \varphi = \text{constans},$$

$$d = \sin \varphi.$$  

In order to determine displacement $\delta_x$ in the direction of axis X with the assistance of Maxwell-Mohr method, the authors used the diagram presented in Figure 4.

The analysis of the adopted loading scheme made it possible to obtain distribution of forces which were appropriate: to the system subjected to the action of the resultant force $P$ as well as individual forces acting in the direction of the sought displacement (Fig. 5).
Applying the above-mentioned energy method of Maxwell-Mohr and after rearranging the equation, the following version of equation $\delta_x$ was obtained:

$$\delta_x = \frac{P}{3E_1}(b^3\sin^2\alpha + a^3\cos^2\varphi)[\text{mm}].$$

(11)

**Displacement $\delta_y$**

Displacement $\delta_y$ was determined with the assistance of the diagram shown in Figure 6 in which the acting force $P$ was replaced by an individual force. Graphical integration with the assistance of the Maxwell-Mohr method was carried out using bending moments from the diagram (Fig. 5a) as well as from the chart of bending moments in the diagram from Figure 7.
Next, an equation of the displacement $\delta_y$ was determined:

$$\delta_y = -\frac{P}{3cE_y} (db^3 \sin \alpha \cos \alpha + la^3 \sin \phi \cos \phi) \text{[mm]}. \quad (12)$$

**Elastic constants**

Having derived equations of $\delta_x$ and $\delta_y$ displacements, the authors proceeded to determine $\varepsilon_x$ and $\varepsilon_y$ deformations in directions of X and Y axes as follows:

$$\varepsilon_x = \frac{P(b^3 \sin^2 \alpha + a^3 \cos^2 \phi)}{3E_x lI}, \quad (13)$$

$$\varepsilon_y = -\frac{P(db^3 \sin \alpha \cos \alpha + la^3 \sin \phi \cos \phi)}{3cE_y lacos \phi}, \quad (14)$$

which, ultimately, allows determination of the Poisson's ratio $\vartheta_{xy}$:

$$\vartheta_{xy} = -\frac{E_x}{E_y} \frac{l(db^3 \sin \alpha \cos \alpha + la^3 \sin \phi \cos \phi)}{c^2(l+\vartheta_{xy})}. \quad (15)$$

On the basis of equations (11) and (12), using the stiffness coefficient $K_i = \frac{P}{\delta_i} \text{[N/mm]}$ and employing the moment of inertia of the cell $I = \frac{gt^3}{12} \text{[mm}^4]$, where: $g$ – height of the core [mm] and $t$ – cell wall thickness [mm], Young modulus $E_x$ was determined:

$$E_x = \frac{4K_x(b^3 \sin^2 \alpha + a^3 \cos^2 \phi)}{gt^3} \text{[MPa]} \quad (16)$$

and $E_y$:

$$E_y = \frac{4K_y(db^3 \sin \alpha \cos \alpha + la^3 \sin \phi \cos \phi)}{cg^t^3} \text{[MPa]}. \quad (17)$$

Later on, after converting the simple dependence into modulus of rigidity $G = \frac{E}{2(1+\vartheta)}$, the following $G_{xy}$ was obtained:

$$G_{xy} = \frac{2K_x(b^3 \sin^2 \alpha + a^3 \cos^2 \phi)}{gt^3(1+\vartheta_{xy})} \text{[MPa]}. \quad (18)$$
2. RESULTS AND DISCUSSION

For the purpose of the performed calculations, a number of assumptions were adopted for the model system:

\[ \alpha = \text{const} = \frac{\pi}{6} \]  \hspace{1cm} (19)
\[ a = \text{const} = 5 \text{ mm} \]  \hspace{1cm} (20)
\[ b = \text{const} = 10 \text{ mm} \]  \hspace{1cm} (21)
\[ l = \text{const} \]  \hspace{1cm} (22)
\[ \beta = \left(\frac{\pi}{3}, 0\right) \]  \hspace{1cm} (23)
\[ \phi = \left(0; \frac{\pi}{3}\right). \]  \hspace{1cm} (24)

From technological point of view, it is advisable to take into consideration changes of elastic properties in the function of the inclination angle of cell walls \( \beta \). Therefore, Figure 8 presents values of the elasticity modulus \( E \) determined for directions \( X \) and \( Y \). Figure 9 illustrates how the modulus of rigidity \( G_{xy} \) altered in the function of angle \( \beta \).

![Figure 8. Change of linear moduli of elasticity \( E_x \) and \( E_y \) in the function of \( \beta \) angle.](image)

Majewski and Smardzewski (2015). "Models for elastic deformation of auxetic honeycomb with triangular cells"
Next, the authors elaborated a variability diagram of the Poisson's ratio $\nu_{xy}$ in the function of the internal inclination angle of $\beta$ arms (Fig. 10). Arbitrarily, it was assumed that the cell material was isotropic.

![Figure 10. Variability of the Poisson's ratio $\nu_{xy}$ in the function of the internal inclination angle of $\beta$ arms.](image)

Figure 8 shows a non-linear course of changes of elastic properties of an auxetic cell. However, in the case of $E_x$, the difference in the obtained values for the set system amounted only to 96.5 MPa. The mean value of 314.6 MPa ($SD = 26.9$ MPa) appears to be in keeping with the expectations resulting from the review of the literature on the subject. It can be concluded that a change in the arm inclination angle of the $E_x$ cell did not cause important alterations in values of the linear elasticity modulus $E_x$. On the other hand, there was a distinct nonlinear change in the value of the $E_y$ modulus when $\beta$ angle was altered. An excessively small inclination angle of arms led to cell slimming which became susceptible to stability loss. It was found that $E_y$ values obtained for the inclination angle of arms $\beta < 15^\circ$ need not be
taken into consideration in further experiments due to the above-mentioned effect. It was
decided that in the context of a potential physical modelling, it is advisable to take into
account values from the interval of $\beta$ angle $< 20^\circ; 40^\circ >$. It was observed that values of the
elasticity modulus decreased together with the increasing inclination angle for both directions
of anisotropy.

Shear modulus $G_{xy}$ also changed in a nonlinear manner for the analysed interval of angle $\beta$.
Judging from the course of the diagram, it can be assumed that the most favourable solution is
to take into consideration values from the interval of the arm inclination angle $\beta = < 20^\circ; 40^\circ >$.

It is evident from Figure 10 that a decrease in the $\beta$ angle caused a nonlinear decline in the
Poisson's ratio. The auxetic effect was best illustrated by geometry in which the internal
inclination angle of arms was less than 10°. However, the designed cell, for technological
reasons, should not be characterised by a value of this range. On the other hand, when the
arms obtuse angle of the cell was large, values of the Poisson's ratio exceeded -2. For that
reason, it was decided to take into consideration intermediate values of the $\beta$ angle contained
within $< 20^\circ; 30^\circ >$.

3. CONCLUSION

The elaborated mathematical model depicts well the deformation character of the triangular
auxetic cell subjected to bending. The diagrams show a nonlinear change of elastic properties
resulting from the presented parameter relationships of cell geometry combined with elastic
properties of the material from which it was manufactured. Bearing in mind potential
possibilities of manufacturing a core with cells of such new structure, it is the value of an
angle $\beta$ that is the most important. It determines production possibilities of a physical model. At a
small $\beta$ angle, paper will rupture during paper tape formation, whereas at a large $\beta$ angle, the
scale of observed displacements caused by the action of the force in direction Y will be small.
This will not exert a favourable impact on the auxetic effect of the analysed system.

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